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PROPERTIES OF INERTIAL-GRAVITATIONAL FIELDS

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ABSTRACT. Certain of the properties of inertial-gravitational fields described by Maxwell type equations are considered.

1. Inertial-gravitational fields are fields that can be described by the /97* equations

$$G^{ik}{}_{;k} = \frac{4\pi\varkappa}{c^4} J^i \tag{1}$$

and

$$G_{ik;l} + G_{li;k} + G_{kl;i} = 0, \tag{2}$$

where c is the speed of light in a vacuum; x is the gravitation constant; J^i are the components of the four-vector current density for the field generator; G_{ik} are the components of the skew-symmetric tensor of rank two with the sense of the four-dimensional intensity of an inertial-gravitational field:

$$G_{ik} \equiv u_{k,i} - u_{i,k}, \tag{3}$$

$u^i = dx^i/ds$ are the potentials of the inertial-gravitational field.

The equations of motion in an inertial-gravitational field are the equations

$$G_{ki} u^k = 0, \tag{4}$$

which are equivalent to the equations for the geodesic

$$u^k u_{i;k} = 0.$$

(Actually, $u_k u^k = -1$, so that $u^k u_{k;i} = 0$ and $(u_{i;k} - u_{k;i})\, u^k = (u_{i,k} - u_{k,i})\, u^k = 0$).

The equations of motion at (4) can be obtained directly from variational principles $\delta S = 0$,

where

$$S = -\frac{1}{c}\int ds = \frac{1}{c}\int u_i dx^i, \tag{5}$$

and $ds = -u_i dx^i$ is an elementary space-time interval.

The variational principles will also provide the equations for the geodesic

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* Numbers in the margin indicate pagination in the foreign text.

in another form:

$$\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = 0 \quad \text{(4a) or} \quad \frac{du_l}{ds} - \Gamma_{k,il} u^k u^l = 0; \quad \text{(4b)}$$

for which the quadratic form

$$ds^2 = -g_{ik} dx^i dx^k.$$

must be used in the expression at (5). The equality

$$-g_{ik} dx^i dx^k = u_i dx^i u_k dx^k, \qquad /98$$

is satisfied, that is $(g_{ik} + u_i u_k) dx^i dx^k = 0.$[1]

Direct transformations (as should be expected) can reduce the equation for the geodesic at (4b) to the equations at (4), and pass from those at (4a) to equations in the form

$$G_k^{\cdot i} \cdot u^k = 0. \quad \text{(4c)}$$

The equations at (4a) (or at (4b)) are form I equations of the geodesic, while those at (4c) (or at (4)) are, correspondingly, form II.

Accordingly, it can be stated that equations of motion in an inertial-gravitational field are form II equations of the geodesic.

There is an obvious analogy between the expressions at (5) and that part of the effect (in electrodynamics) resulting from the interaction between the charge and the electromagnetic field [3]. The inertial-gravitational field tensor G_{ik} at (3) is similar to the electromagnetic field tensor $F_{ik} = A_{k,i} - A_{i,k}$ (A_i are electromagnetic potentials). Thus, each space-time point can have attributed to it four functions of $u_i(x)$, determined with an accuracy within the transformations permissible for them, similar to the way in which electromagnetic potentials can be introduced in electrodynamics. The magnitudes u_i play the role of inertial-gravitational field potentials. They acquire as well the sense of

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1. Some functions φ_{ik} such that $g_{ik} + u_i u_k = \varphi_{ik}$ can be introduced (see [2] for example). This function should satisfy the condition that follows directly from the written relationship, specifically, $\varphi_{ik} u^k = 0$.

four-speed components when test particles move in the field. This duality of the functions of inertial-gravitational potentials is one of the fundamental properties of the inertial gravitational field.

The inertial-gravitational field equations at (1) too can be obtained from variational principles.

If it is taken that there are no other material objects in space-time other than the inertial-gravitational field and its sources - masses, the motions of which are given, then the Lagrangian density can be represented in the form

$$\Lambda = \Lambda_f + \Lambda_m,$$

where Λ_f is the Lagrangian density of the field, constructed from the components of the inertial-gravitational field tensor

$$\Lambda_f = -\frac{c^4}{16\pi\varkappa} G^{ik} G_{ik},$$

Λ_m is the density of the Lagrangian function of field sources

$$\Lambda_m = J^i u_i.$$

The effect of S_{mf}, variations in which will only be subject to magnitudes characterizing the field, is consequently, in the form [2]

$$S_{mf} = \frac{1}{c} \int \left(J^i u_i - \frac{c^4}{16\pi\varkappa} G_{ik} G^{ik} \right) \sqrt{-g}\, d\Omega, \tag{6}$$

where g is a determinant composed of the components of the metric tensor g_{ik}, $d\Omega = dx^0 dx^1 dx^2 dx^3$.

The equations at (1) can be obtained from the least effect principle /99

$$\delta S_{mf} = 0. \tag{7}$$

But at this point it is important to take note of the following. If the equations at (1) are multiplied by u_i, we obtain

$$(G^{ik} u_i)_{;k} - G^{ik} u_{i;k} = \frac{4\pi\varkappa}{c^4} J^i u_i.$$

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2. The current density J^i will be unchanged in this particular case because the motion of the source is given, as has already been pointed out.

But the first summand in the left side of this equation equals zero because $G^{ik}u_i = 0$ (4), while the second can be reduced very readily to the form

$$G^{ik}u_{i;k} = -\frac{1}{2}G^{ik}G_{ik},$$

so that

$$J^l u_i = \frac{c^4}{8\pi\varkappa}G^{ik}G_{ik}. \tag{8}$$

Comparing this equality with the expression under the integral sign at (6), we see that the equations at (1) can be obtained from the least effect principle at (7) in the form

$$G^{ik}_{;k} = 0, \tag{1a}$$

that is

$$J^l = 0. \tag{9}$$

Thus, the current density for the cource of an inertial-gravitational field when only this field and the gravitating masses are present equals zero. As will be seen from the preceding analysis, this fact is directly associated with the fact that the equations of free motion in an inertial-gravitational field are form II equations of a geodesic.

Since equating the four-dimensional vector for the denisty of an inertial-gravitational field current to zero, as at (9), does not mean there is no gravitating matter, as follows from what has been said, we will assume that a four-density inertial-gravitational current can be represented in the form of the difference in densities between real gravitational and "inertial" currents. From this point of view the expression at (9) must be considered a statement of the fact that when only an inertial-gravitational field and gravitating matter exist, the four-dimensional vectors for the densities of the "inertial" and real gravitational currents are equal. This, obviously, is fully explained by the identity of the inertial and gravitating masses. Hence

$$J^l = J^l_{(i)} - J^l_{(g)},$$

where $J^i_{(i)}$ and $J^i_{(g)}$ are the components of the four-vectors for the densities of the "inertial" and real gravitational currents, respectively.

But the densities of the "inertial" and the "gravitational" currents are equal

$$J^l_{(i)} = J^l_{(g)} \tag{9a}$$

only when there are no material objects other than the gravitating masses and the inertial-gravitational field in the space-time domain under consideration.

In the general case, however, the relationships at (9) and (9a) cannot occur (because additional terms appear in the expression at (6) for the S_{mf} effect).

The equations at (1) can now be rewritten as /100

$$G^{lk}{}_{;k} = \frac{4\pi\varkappa}{c^4}(J^l_{(i)} - J^l_{(g)}) \tag{1b}$$

or, in expanded form in covariant components

$$u^k{}_{;l;k} - u_l{}^{;k}{}_{;k} = \frac{4\pi\varkappa}{c^4}(J^l_{i(i)} - J^l_{i(g)}).$$

Adding and subtracting the $u^k_{;k;i}$ terms in the left side of the equations written, and noting that

$$u^k{}_{;l;k} - u^k{}_{;k;l} = R_{lk}u^k$$

(R_{ik} are the components of the Richey tensor), we obtain [4].

$$u_l{}^{;k}{}_{;k} - u^k{}_{;k;l} - R_{lk}u^k = -\frac{4\pi\varkappa}{c^4}(J_{l_{(i)}} - J_{l(g)}). \tag{1c}$$

2. If, in accordance with the general theory of relativity, it is taken that the presence of the components, R_{ik}, of the Richey tensor different from the zero, means the existence of a gravitational field, it is natural to suppose that equations for the strictly gravitational field

$$-R_{lk}u^k = \frac{4\pi\varkappa}{c^4}J_{l(g)},$$

can be chosen from the equations for an inertial-gravitational field at (1c). The density of the strictly gravitational current $J_{i(g)}$ can be expressed by some tensor τ_{ik} which is associated with the tensor for the energy pulse of the gravitating system $T_{ik_{(m)}}$:

$$J_{l(g)} = -\tau_{lk}u^k.$$

We have

$$R_{lk}u^k = \frac{4\pi\varkappa}{c^4}\tau_{lk}u^k. \tag{10}$$

If the requirement that the field equations not contain potentials is clear (in particular case u^k), the equations at (10) must retain their validity for any u^k, and this is equivalent to the assertion that

$$R_{lk} = \frac{4\pi\varkappa}{c^4}\tau_{lk}. \tag{11}$$

When $\tau_{lk} = 2\left(T_{lk_{(m)}} - \frac{1}{2}g_{lk}T_{(m)}\right)$, the equations at (11) coincide with those postulated by the general theory of relativity as gravitational field equations.

It is very curious that the potentials of the inertial-gravitational field, u_i, usually play no part in the equations at (11). Here the potentials are the components of the metric tensor g_{ik} associated with the magnitudes u_i by the normalization conditions $g_{ik}u^iu^k = -1$.

3. The energy tensor $T^i_{k(f)}$ can be established for an inertial-gravitational field.

The equations at (2), since they are multiplied by G^{ik}, become, after simple transformations

$$\frac{1}{a}T^m_{l;m(f)} = -G_{ll}G^{lm}{}_{;m},$$

where the magnitudes /101

$$T^m_{l(f)} = a\left(G^{im}G_{ll} - \frac{1}{4}\delta^m_l G^{ik}G_{ik}\right) \tag{12}$$

form the energy tensor for an inertial-gravitational field (a is some constant, δ^m_l is Kronecker's delta symbol). The inertial-gravitational field energy tensor can be found using other, known, methods.

In the special case when the density of the inertial-gravitational current is equal to zero, as in the case of the field described by the equations at (1a) and (2), the expression for the inertial-gravitational field energy tensor can be simplified into

$$T^m_{l(f)} = aG^{mn}G_{ln}$$

($G^{ik}{}_{;k} = 0$, $G^{ik}u_k = 0$, and, consequently, $G^{ik}G_{ik} = 0$).

4. Since the determination of the inertial-gravitational field tensor G_{ik} (3) is analogous to the determination of the electromagnetic field tensor F_{ik} through potentials A_i, and since the inertial-gravitational field equations at

(1) and (2) are Maxwell type equations, it is clear that the inertial-gravitational field equations are invariant with respect to the gradient transformations of field potentials

$$u_i' = u_i + f_{,i}, \tag{13}$$

where f is a scalar function of the coordinates.

But attention must be given to the significant difference between the transformations of inertial-gravitational field potentials u_i (13) and the analogous procedure in electrodynamics wherein normalization of field potentials is unknown, while inertial-gravitational potentials are normalized

$$u_i u^i = u_i' u^{i'} = \cdots = u_i^{(n)} u^{i(n)} = -1,$$

so the function f should therefore satisfy the relationship

$$2\frac{df}{ds} + f_{,i} f^{,i} = 0. \tag{14}$$

It is now of interest to discuss the possibility of applying the limitations analogous to Lorentz conditions used in electrodyanmics, $A^i_{;i} = 0$, to the inertial-gravitational field potentials u^i.

In arriving at the solution to this problem, let us avail ourselves of the fact that the u magnitudes carry what is, in a sense, a dual load. Because they are potentials of an inertial-gravitational field they can play the role of the components of four-speed test particles moving in the field.

Let us visualize the continuum of identical test particles that are not interacting with each other (an incoherent fluid) moving in an inertial-gravitational field. Designating the invariant density of the incoherent fluid under consideration by μ, let us write the equation of continuity for it in general covariant form

$$j^i{}_{;i} = 0,$$

where $j^i = \mu c^2 u^i$, that is

$$\mu c^2 u^i{}_{;i} + c^2 u^i \mu_{,i} = 0.$$

This means that the divergence of the four-speed (four-vector potential) is different from zero. /102

$$u^i{}_{;i} = -\frac{1}{\mu}\frac{d\mu}{ds} \neq 0. \tag{15}$$

In order to satisfy the Lorentz conditions, let us transform the potentials u^i such that the divergences of the new potentials, $u^{(0)i}$, equal zero

$$u^{(0)l}{}_{;l} = 0;$$

here

$$u^{(0)l} = u^l + f^{(0),l}.$$

Thus, the equality should be satisfied

$$u^l{}_{;l} + f^{(0),l}{}_{;l} = 0.$$

Or, if the equation at (15) is taken into consideration, the function $f^{(0)}$ should be such that

$$f^{(0),l}{}_{;l} - \frac{1}{\mu}\frac{d\mu}{ds} = 0. \tag{16}$$

In other words, the Lorentz conditions for inertial-gravitational field potentials can be satisfied if the potentials can be transformed as in (13) so as to simultaneously satisfy the relationships at (14) and (16). But let us note that (16) assumes a known arbitrariness for the magnitude $f^{(0),i}$, for example:

$$f^{(0),l} = f'^{,l} + \omega^{kl}{}_{;k},$$

where ω^{ki} is any skew-symmetric tensor of rank two.

5. A specific feature of inertial-gravitational fields is that these fields do not always occur where masses exist. There are forms of space-time distribution of matter (mass) when inertial-gravitational fields will not be generated.

A static mass with central symmetry cannot create an inertial-gravitational field, for example.

If the "four-spherical" coordinates r, θ, φ, ct, are introduced and denote x^1, x^2, x^3, x^0, respectively, we obtain

$$u_0 = e, \qquad u_2 = f, \qquad u_3 = h,$$

where e, f, h are some constants, directly from the equations of motion at (4), taking into consideration the fact that the centrally symmetrical field under consideration is static.

Thus, it develops that three of the four magnitudes determining the four-vector potential of an inertial-gravitational field are constants. In so far as the u_1 components are concerned, we note that only $u_{1,1}$ of the derivatives of $u_{1,i}$ can be different from zero in this case. But this magnitude is only

contained in the $G_{11(cs)}$ component of the inertial-gravitational field tensor identically equal to zero.

Consequently, all components of the four-dimensional "intensity" of a static inertial-gravitational field with central symmetry equal zero, $G_{ik(cs)} = 0$. There is no field.

6. Let us consider the motion of an incoherent fluid carrying an electrical charge in inertial-gravitational and electromagnetic fields.

The equations of motion can be represented in the form /103

$$\mu c^2 u_{l;l} u^l = \rho F_{ll} u^l$$

(ρ is the invariant charge density). These equations are the equations for the lines of current in an incoherent charged fluid.

The term $u_{l;i}u^l$, when multiplied by μc^2, equals zero and is subtracted from the left side of these equations to obtain

$$\mu c^2 (u_{l,l} - u_{l,l}) u^l = \rho F_{ll} u^l.$$

or, using the determination at (3)

$$(\mu c^2 G_{ll} + \rho F_{ll}) u^l = 0. \tag{17}$$

The equation at (17) can also be written in the form

$$\rho W_{ll} u^l = 0, \tag{17a}$$

where $W_{ll} = \frac{1}{\varepsilon} G_{ll} + F_{ll} = \Pi_{l,l} - \Pi_{l,l}$, and $\Pi_l = \frac{1}{\varepsilon} u_l + A_l$ — are components of the "generalized" four-vector potential $\varepsilon = \frac{\rho}{\mu c^2} =$ constant.

What follows from the determination of the skew-symmetric tensor of rank two, W_{li}, as an identity are the equations $W_{lk;l} + W_{ll;k} + W_{kl;l} = 0$, which, after multiplication by W^{ik}, yield $\Theta^k_{l;k} = -W_{ll} \cdot W^{lk}{}_{;k}$,

where $\Theta^k_l = W^{km}W_{lm} - \frac{1}{4}\delta^k_l W^{mn}W_{mn} = t^k_{l(l)} + t^k_{l(e)} + t^k_{l(le)}$, $t^k_{l(e)} = F^{km}F_{lm} - \frac{1}{4}\delta^k_l F^{mn}F_{mn}$ are the components of the electromagnetic field energy tensor;

$t^k_{l(l)} = \frac{1}{\varepsilon^2}\left(G^{km}G_{lm} - \frac{1}{4}\delta^k_l G^{mn}G_{mn}\right)$ are the components of the tensor determining the

energy contribution of the inertial-gravitational field to the interaction of the fields with the test particles.

The approach to the last summand requires known care because it does not determine the direct interaction of the fields with each other.

$t^k_{l(ie)} = \frac{1}{\varepsilon}\left(2G^{km}F_{lm} - \frac{1}{2}\delta^k_l G^{mn}F_{mn} \right)$ can be called the energy tensor taking the interaction between the "test particle" electromagnetic and inertial-gravitational fields into consideration.

7. The analysis made of certain of the properties of inertial-gravitational fields described by the Maxwell type equations at (1) and (2) reveals that the term "inertial-gravitational field" is understood to mean a physical situation differing, in the general case, from what one has in mind when only real gravitational fields described by the equations of the general theory of relativity are considered.

A significant, special feature of the inertial-gravitational field is that its potentials, u, determine the field of the four-vector speed of test particles.

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